International Journal of Theoretical Physics, Vol. 34, No. 1, 1995

Existence of Exponential Dichotomy of a Class of Impulsive Differential Equations

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Received March 13, 1992

Sufficient conditions for the existence of an exponential dichotomy of linear impulsive differential equations with almost periodic operator-valued function are found.

1. INTRODUCTION

In the present paper the investigations of Bainov *et al.* (1989) are continued. A dependence between the existence of an exponential dichotomy and the existence of bounded solutions of the corresponding nonhomogeneous equation is established. Similar investigations of equations without an impulse effect have been carried out in Sacker and Sell (1974), Coppel (1978), and Palmer (1984).

2. STATEMENT OF THE PROBLEM

Let X be a finite-dimensional Banach space with identical operator I. By J we shall denote $\mathbb{R}$ or $\mathbb{R}_+ = [0, \infty)$, and by Ω, we denote $\mathbb{Z}$ or $\mathbb{N} \cup \{0\}$. By $L(X)$ we denote the space of all linear bounded operators which act in X.

Consider the homogeneous

$$\frac{dx}{dt} = A(t)x \qquad (t \neq t_n) \tag{1}$$

$$x(t^+) = Q(t)x \tag{2}$$

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and nonhomogeneous

$$\frac{dx}{dt} = A(t)x + f(t) \qquad (t \neq t_n) \tag{3}$$

$$x(t^+) = Qx(t) + h(t) \tag{4}$$

impulsive differential equations, where $x: J \to X$, $A(\cdot): J \to L(X)$ is a continuous and bounded function, the function $Q(\cdot)$ has the form

$$Q(t) = \sum_{n \in \Omega} Q_n \delta(t - t_n)$$

$[Q_n, \; Q_n^{-1} \in L(X), n \in \Omega]$, $\delta(t)$ is the Dirac function, $f: J \to X$ is a given continuous function, and $h(t)$ has the form

$$h(t) = \sum_{n=1}^{\infty} h_n \delta(t - t_n) \qquad (h_n \in X, n \in \Omega)$$

The sequence of points $\{t_n\}$ satisfies the conditions $t_n < t_{n+1}$ $(n \in \Omega)$ and $\lim_{n \to \infty} t_n = \infty$ (respectively, $\lim_{n \to \pm\infty} t_n = \pm\infty$).

Remark 1. Equations (2) and (4) can be written down in a standard form without the δ-function, as follows:

$$x(t_n^+) = Q_n x(t_n) \qquad (n \in \Omega)$$

$$x(t_n^+) = Q_n x(t_n) + h \qquad (n \in \Omega)$$

Definition 1. The impulsive equation (1), (2) is said to be *exponentially dichotomous* if there exist projectors $P_1, P_2: X \to X$ $(P_1 + P_2 = 1)$ and constants $K, \kappa > 0$ for which the inequalities

$$\| V(t) P_1 V^{-1}(s) \| \leq K e^{-\kappa(t-s)} \qquad (s \leq t; s, t \in J) \tag{5}$$

$$\| V(t) P_2 V^{-1}(s) \| \leq K e^{-\kappa(s-t)} \qquad (t \leq s; s, t \in J) \tag{6}$$

are valid, where $V(t)$ is the evolutionary operator of equation (1), (2) for which $V(0) = I$ (Bainov *et al.*, 1989).

Lemma 1. Let the number $h > 0$ be arbitrarily chosen.
Then the impulsive equation

$$\frac{dx}{dt} = A(t+h)x \qquad (t \neq t_n) \tag{7}$$

$$x(t^+) = Q(t+h)x \tag{8}$$

has an evolutionary operator $V_h(t)$ which satisfies the condition

$$V_h(t) = V(t+h) V^{-1}(h)$$

The proof of Lemma 1 follows from the definition of the evolutionary operator $V(t)$ and from the fact that the solution $x(t)$ of equation (1), (2) and the solution $x_h(t)$ of (7), (8) are related by the equality

$$x(t-h) = x_h(t)$$

Consider the impulsive equations

$$\frac{dx}{dt} = A^{(1)}(t)x \qquad (t \neq t_n) \tag{9}$$

$$x(t_n^+) = Q^{(1)}(t)x(t) \tag{10}$$

and

$$\frac{dx}{dt} = A^{(2)}(t)x \qquad (t \neq t_n) \tag{11}$$

$$x(t_n^+) = Q^{(2)}(t)x(t) \tag{12}$$

with evolutionary operators $W_1(t)$ and $W_2(t)$, respectively.

Lemma 2. For $t, s \in \mathbb{R}$ let the following inequality be valid:

$$\| W_1(t,s) \| \leq N e^{-v(h(t)-h(s))} \tag{13}$$

where $h(t)$ $(t \in \mathbb{R})$ is an arbitrary function for which $h(0) = 0$, and let the operators $Q_n^{(1)}$ and $Q_n^{(2)}$ have bounded inverse ones.

Then the estimates

$$\| W_2(t,s) \| \leq N\left(\exp\{-v[h(t)-h(s)]\} \exp\left[N \int_s^t p(\tau)\, d\tau \right]\right) \times \prod_{s<t_j<t} [1+q(t_j)] \tag{14}$$

and

$$\begin{aligned} &\| W_2(t,0) - W_1(t,0) \| \\ &\quad \leq N\{\exp[-vh(t)]\}\Delta(t)\left\{\exp\left[N \int_0^t p(s)\, ds \right] - 1\right\} \\ &\qquad + N\{\exp[-vh(t)]\} \sum_{0<t_j<t} Nq(t_j)\left\{\exp\left[N \int_0^{t_j} p(\tau)\, d\tau\right]\right\} \\ &\qquad \times \prod_{0<t_k<t_j} [1+q(t_k)] \end{aligned} \tag{15}$$

are valid, where

$$q(t) = \begin{cases} 0, & t \neq t_n \\ \| Q_n^{(1)} - Q_n^{(2)} \|, & t = t_n \end{cases}$$

$$\Delta(t) = \prod_{0<t_i<t} [1+q(t_i)], \qquad p(t) = \| A^{(2)}(t) - A^{(1)}(t) \|$$

Proof. The operator $V_2(t) = W_2(t, 0)$ is a solution of the operator-impulsive equation

$$\frac{dV_2}{dt} - A^{(1)}W_2 = (A^{(2)} - A^{(1)})W_2 \qquad (t \neq t_n) \tag{16}$$

$$V_2(t_n^+) = Q^{(1)}(t)V_2(t) + [Q^{(2)}(t) - Q^{(1)}(t)]V_2(t) \tag{17}$$

$$V_2(0) = I$$

From the definition of the operator $W_1(t, s)$ it follows that for the solution $X(t)$ of equation (16), (17) we obtain the formula

$$X(t) = W_1(t, 0) + \int_0^t W_1(t, \tau)[A^{(2)}(\tau) - A^{(1)}(\tau)]V_2(\tau)\,d\tau$$

$$+ \sum_{0<t_j<t} W_1(t, t_j^+)(Q_j^{(2)} - Q_j^{(1)})V_2(t_j) \tag{18}$$

From equality (18) there follows the estimate

$$\varphi(t) \leq N e^{-\nu h(t)} + N \int_0^t e^{-\nu(h(t)-h(s))}p(\tau)\varphi(\tau)\,d\tau$$

$$+ \sum_{0<t_j<t} e^{-\nu(t-t_j)}q(t_j)\varphi(t_j) \tag{19}$$

where $\varphi(t) = \| V_2(t) \|$. Set $\varphi(t) = \varphi_1(t)\,e^{-\nu h(t)}$. From inequality (19) for $\varphi_1(t)$ we obtain

$$\varphi_1(t) \leq N + N \int_0^t \varphi_1(\tau)p(\tau)\,d\tau + \sum_{0<t_j<t} q(t_j)\varphi_1(t_j)$$

From Samoilenko and Perestyuk (1987) there follows the estimate

$$\varphi_1(t) \leq N\left\{\exp\left[N \int_0^t p(\tau)\,d\tau\right]\right\} \prod_{0<t_j<t} [1 + q(t_j)]$$

hence

$$\varphi(t) \leq N\left\{\exp[-\nu h(t)]\,\exp\left[N \int_0^t p(\tau)\,d\tau\right]\right\} \prod_{0<t_j<t} [1 + q(t_j)]$$

Equality (14) is proved. We shall prove (15). Equality (18) implies

$$V_2(t) = W_1(t, 0) + \int_0^t W_1(t, \tau)[A^{(2)}(\tau) - A^{(1)}(\tau)]V_2(\tau)\,d\tau$$

$$+ \sum_{0<t_j<t} W_1(t, t_j^+)(Q_j^{(2)} - Q_j^{(1)})V_2(t_j)$$

hence

$$\begin{aligned}
&\|V_2(t)-V_1(t)\| \\
&\leq \int_0^t N(\exp\{-v[h(t)-h(\tau)]\}p(\tau)N\left\{\exp[-vh(\tau)]\exp\left[N\int_0^\tau p(s)\,ds\right]\right\} \\
&\quad\times\prod_{0<t_j<\tau}[1+q(t_j)]\,d\tau+\sum_{0<t_j<t}N\{\exp-v[h(t)-h(t_j)]\}q(t_j) \\
&\quad\times N\left\{\exp[-vh(t_j)]\exp\left[N\int_0^{t_j}p(\tau)\,d\tau\right]\right\}\prod_{0<t_k<t_j}[1+q(t_k)] \\
&=N\{\exp[-vh(t)]\}\int_0^t Np(\tau)\left\{\exp\left[N\int_0^\tau p(s)\,ds\right]\right\}\prod_{0<t_j<\tau}[1+q(t_j)]\,d\tau \\
&\quad+N\{\exp[-vh(t)]\}\sum_{0<t_j<t}Nq(t_j)\left\{\exp\left[N\int_0^{t_j}p(\tau)\,d\tau\right]\right\} \\
&\quad\times\prod_{0<t_k<t_j}[1+q(t_k)] \qquad (20)
\end{aligned}$$

Taking into account the equality

$$\int_a^b Np(\tau)\exp\left[N\int_0^\tau p(s)\,ds\right]d\tau=\exp\left[N\int_0^a p(s)\,ds\right]-\exp\left[N\int_0^a p(s)\,ds\right]$$

and setting $t_0=0$, for the first addend in (20) we obtain the estimate

$$\begin{aligned}
&N\{\exp[-vh(t)]\}\int_0^t Np(\tau)\left\{\exp\left[N\int_0^\tau p(s)\,ds\right]\right\}\prod_{0<t_j<\tau}[1+q(t_j)]\,d\tau \\
&\quad=N\{\exp[-vh(t)]\}\sum_{j=1}^{n(t)}\left\{\int_{t_{j-1}}^{t_j}Np(\tau)\left\{\exp\left[N\int_0^\tau p(s)\,ds\right]\right\}\right. \\
&\qquad\left.\times\prod_{0<t_i<\tau}[1+q(t_i)]\,d\tau\right\}+N\exp[-vh(t)] \\
&\qquad\times\int_{t_{n(t)}}^t Np(\tau)\left\{\exp\left[N\int_0^\tau p(s)\,ds\right]\right\}\prod_{0<t_j<\tau}[1+q(t_j)]\,d\tau \\
&\quad\leq N\{\exp[-vh(t)]\}\prod_{0<t_i<t}[1+q(t_i)]\sum_{j=1}^{n(t)}\left\{\exp\left[N\int_0^{t_j}p(s)\,ds\right]\right. \\
&\qquad\left.-\exp\left[N\int_0^{t_{j-1}}p(s)\,ds]\right]\right\}+N\{\exp[-vh(t)]\} \\
&\qquad\times\prod_{0<t_i<t}[1+q(t_i)]\left\{\exp\left[N\int_0^t p(s)\,ds\right]-\exp\left[N\int_0^{t_{n(t)}}p(s)\,ds\right]\right\} \\
&\quad\leq N\{\exp[-vh(t)]\}\Delta(t)\left\{\exp\left[N\int_0^t p(s)\,ds\right]-1\right\}
\end{aligned}$$

Lemma 2 is proved. ■

Lemma 3. Let the following conditions hold:

1. Equation (1), (2) is exponentially dichotomous on $\mathbb{R}_+$.
2. There exists a sequence of integers $\{h_\nu\}_{\nu=1}^{\infty}$, $h_\nu \to \infty$, such that $\lim_{\nu\to\infty} A(t+h_\nu) = B(t)$ and $\lim_{\nu\to\infty} Q(t+h_\nu) = \tilde{Q}(t)$ uniformly on each compact subinterval of $\mathbb{R}$.

Then:

1. There exists a projector $\tilde{P}$: $X \to X$ for which

$$\lim_{\nu\to\infty} V(h_\nu)P_1 V^{-1}(h_\nu) = \tilde{P}$$

2. The impulsive equation

$$\frac{dy}{dt} = B(t)y \qquad (t \neq t_n) \tag{21}$$

$$y(t^+) = \tilde{Q}(t)y(t) \tag{22}$$

is exponentially dichotomous on $\mathbb{R}$ with projectors $\tilde{P}$ and $I - \tilde{P}$.

Proof. Consider the impulsive equation

$$\frac{dx}{dt} = A(t+h_\nu)x \qquad (t \neq t_n)$$

$$x(t^+) = Q(t+h_\nu)x(t)$$

with an evolutionary operator $V_\nu(t)$, which by Lemma 1 satisfies the condition $V_\nu(t) = V(t+h_\nu)V^{-1}(h_\nu)$, where $V(s)$ is the evolutionary operator of (1), (2). Set $P_\nu = V(h_\nu)P_1 V^{-1}(h_\nu)$. For $\|V_\nu(t)P_\nu V_\nu^{-1}(s)\|$ we obtain the estimate

$$\begin{aligned}
&\|V_\nu(t)P_\nu V_\nu^{-1}(s)\| \\
&\quad = \|V_\nu(t)V(h_\nu)P_1 V^{-1}(h_\nu)V_\nu^{-1}(s)\| \\
&\quad = \|V(t+h_\nu)V^{-1}(h_\nu)V(h_\nu)P_1 V^{-1}(h_\nu)V(h_\nu)V^{-1}(s+h_\nu)\| \\
&\quad = \|V(t+h_\nu)P_1 V^{-1}(s+h_\nu)\| \le K e^{-\kappa(t-s)} \qquad (-h_\nu \le s \le t < \infty)
\end{aligned} \tag{23}$$

and, analogously,

$$\|V_\nu(t)(I-P_\nu)V_\nu^{-1}(s)\| \le K e^{-\kappa(s-t)} \qquad (-h_\nu \le t \le s < \infty) \tag{24}$$

For $\nu \in \mathbb{N}$ from (23) it follows that $\|P_\nu\| \le K$. There exists a subsequence $\{\nu_k\}$ for which there exists the limit $\tilde{P} = \lim_{k\to\infty} P_{\nu_k}$. It is trivial to check that $\tilde{P}$ is a projector.

Let $W(t)$ be the evolutionary operator of the impulsive equation (21), (22) for which $W(0) = I$. From Lemma 2 it follows that for

$t \in \mathbb{R}$, $\lim_{\nu \to \infty} V_\nu(t) = W(t)$. Then from (23) as $\nu_k \to \infty$ we obtain the estimate

$$\| W(t)\tilde{P}W^{-1}(s) \| \leq K e^{\kappa(t-s)} \qquad (-\infty < s \leq t < \infty) \tag{25}$$

Analogously, (24) implies the estimate

$$\| W(t)(I - \tilde{P})W^{-1}(s) \| \leq K e^{\kappa(s-t)} \qquad (-\infty < t \leq s < \infty) \tag{26}$$

Lemma 3 is proved. ■

Definition 2. The function α: $\mathbb{R} \to Y$ (Y is an arbitrary Banach space) is said to be *almost periodic* if each sequence $\{h_\nu\} \subset \mathbb{R}$ contains a subsequence $\{h_{\nu_k}\}$ such that uniformly for $T \in \mathbb{R}$ there exists the limit $\lim_{k \to \infty} \alpha(t + h_{\nu_k})$.

Remark. The function α is almost periodic if and only if for any $\varepsilon > 0$ there exists a number $l(\varepsilon)$ such that for each interval of length $l(\varepsilon)$ there exists a number ω for which

$$\| \alpha(t + \omega) - \alpha(t) \| < \varepsilon \qquad (t \in \mathbb{R})$$

3. MAIN RESULTS

Theorem 1. Let the following conditions hold:

1. The conditions of Lemma 3 are met.
2. The functions $A(t)$ and $Q(t)$ are almost periodic on $\mathbb{R}$.

Then the impulsive equation (1), (2) is exponentially dichotomous on $\mathbb{R}$.

Proof. From condition 2 of Theorem 1 it follows that there exists a sequence $h_\nu \to \infty$ for which $A(t + h_\nu)$ uniformly tends to $A(t)$ and $Q(t + h_\nu)$ uniformly tends to $Q(t)$. The assertion of the theorem follows from Lemma 3.

Theorem 1 is proved. ■

Theorem 2. Let the following conditions hold:

1. The nonhomogeneous equation (3), (4) for $h(t) \equiv 0$ has a bounded on $\mathbb{R}_+$ solution for each function f: $\mathbb{R}_+ \to X$ which is a restriction of an almost periodic function on $\mathbb{R}$.
2. $\sup_n \| Q_n^{-1} \| < \infty$.
3. There exists a number $l > 0$ and an integer λ such that each interval of length l contains not more than λ members of the sequence $\{t_n\}$.

Then the homogeneous impulsive equation (1), (2) is exponentially dichotomous on $\mathbb{R}_+$.

Proof. Denote by $C'(\mathbb{R}_+, X)$ the set of all functions $g: \mathbb{R}_+ \to X$ which are continuous for $t \neq t_n$, have discontinuities of the first kind at $t \neq t_n$, and are bounded. With respect to the norm

$$\|g\|_C = \sup_{0 \le t < \infty} \|g(t)\|$$

the set $C(\mathbb{R}_+, X)$ is a Banach space.

By Theorem 2 (Bainov *et al.*, 1989) it suffices to show that the nonhomogeneous equation (3), (4) has a solution in $C(\mathbb{R}_+, X)$ for each function $f \in C(\mathbb{R}_+, X)$ and $h = 0$. Let $\tilde{A}$ be the set of all functions $f(t)$ defined on $\mathbb{R}_+$ which are restrictions of almost periodic functions defined on $\mathbb{R}$. The set $\tilde{A}$ is closed in $C(\mathbb{R}_+, X)$, hence it is a subspace of $C(\mathbb{R}_+, X)$. From Theorem 2 (Bainov *et al.*, 1989) it follows that there exists a constant k such that for each function $f \in \tilde{A}$ the nonhomogeneous equation (3), (4) has a solution $y \in C(\mathbb{R}_+, X)$ satisfying the estimate

$$\|y\|_C \le k\|f\|_C$$

Let $g \in C(\mathbb{R}_+, X)$ and $T_0 > 0$ be arbitrarily chosen. For any $\omega \in (0, 2\pi/T)$ there exists a piecewise continuous function $f(t)$ defined on $\mathbb{R}$ with period $2\pi/\omega > T_0$, which coincides with $g(t)$ on $[0, T_0]$ and for which $\|f\|_C \le \|g\|_C$. Since $f \in \tilde{A}$, then there exists a solution y_f of the impulsive equation

$$\frac{dy}{dt} = A(t)y + f(t) \qquad (t \neq t_n)$$

$$y(t^+) = Q(t)y(t)$$

for which $\|y_f\|_C \le k\|g\|_C$.

Let $\{T_\nu\}$ be an arbitrary sequence for which $\lim_{\nu\to\infty} T_\nu = \infty$. As a result of the above arguments we obtain a sequence of periodic functions $\{f_\nu(t)\}$ as well as a sequence $\{y_\nu(t)\}$ of bounded solutions of the equations

$$\frac{dy}{dt} = A(t)y + f_\nu(t) \qquad (t \neq t_n)$$

$$y(t^+) = Q(t)y(t)$$

for which $\|y_\nu\|_C \le r\|g\|_C$. The sequence $\{y_\nu(0)\}$ is bounded. There exists a subsequence $\{y_{\nu_k}(0)\}$ for which $\lim_{k\to\infty} y_{\nu_k}(0) = \eta$. From the construction of the function $f_\nu(t)$ it follows that $f_{\nu_k}(t) \to_{k\to\infty} g(t)$ uniformly on each compact subinterval of $\mathbb{R}_+$, which implies that $y_{\nu_k}(t) \to_{k\to\infty} y(t)$ for each t and $y(t)$ is a solution of the equation

$$\frac{dy}{dt} = A(t)y + g(t) \qquad (t \neq t_n)$$

$$y(t^+) = Q(t)y(t)$$

for which $y(0) = \eta$. It is not hard to check that the function $y(t)$ is bounded for $t \geq 0$.

Theorem 2 is proved. ■

ACKNOWLEDGMENT

This work was supported by the Bulgarian Ministry of Science and Education under Grant MM-7.

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